VOLUME 78 SEPARATE No. 136

PROCEEDINGS

AMERICAN SOCIETY OF CIVIL ENGINEERS

JUNE, 1952

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

STEADY-STATE FORCED VIBRATION OF CONTINUOUS FRAMES

By C. T. G. Looney, A. M. ASCE

STRUCTURAL DIVISION

Headquarters of the Society
33 W. 39th St.
New York 18, N.Y.

PRICE $0.50 PER COPY

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Proceedings—Vol. 78 June, 1952 Separate No. 136

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

STEADY-STATE FORCED VIBRATION OF CONTINUOUS FRAMES

By C. T. G. Looney,[1] A. M. ASCE

Synopsis

The analysis of the vibration of continuous frames subjected to a periodic force is important in the study of the design of structures that support machinery. The motion of the machinery produces a force that is usually small, but which, because of its periodic nature, sometimes causes large deflections to develop. Analyses have been made of the forced vibrations of simple systems —that is, a spring supporting a mass, and a simply supported beam. The study of simply supported beams by means of harmonic analysis is particularly congruous in the relation of the mathematics to the nature of the physical phenomena.

The theory presented in this paper makes it possible to analyze continuous frames, utilizing the system of harmonic analysis as it is applied to simple beams. The structure will here be treated as a series of simply supported members with periodic end moments which will provide continuity.

1. Theory

The application of this analysis to continuous frames reveals an interesting phenomenon. In order to attain the greatest deflection due to a periodic force, the continuous frame must behave as a series of simply supported members with very small end moments. The deflection of each member must be, very closely, the shape of one of the normal modes of a simply supported beam. Therefore, the proposed method is particularly adapted to the analysis of these conditions.

It will first be necessary to describe the vibration of a simply supported member due to a uniformly distributed or a concentrated periodic force, with special attention to the determination of the end slopes. This will be followed

Note.—Written comments are invited for publication; the last discussion should be submitted by December 1, 1952.

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by an analysis of the vibration of a simply supported beam due to a periodic end moment, and finally, a description of the method used to determine the magnitude and phase of the end moments required to provide continuity.

The equation for the forced vibration of a beam with damping proportional to the velocity is

$$E\,I\,\frac{d^4y}{dx^4} + m\,\frac{d^2y}{dt^2} + 4\,\pi\,\delta\,m\,\frac{dy}{dt} = w \qquad (1)$$

in which E is the modulus of elasticity; I is the rectangular moment of inertia; m is the mass per unit of length; δ is the damping constant, proportional to the velocity; y is the deflection of the beam at a distance x from the left end (y is positive downward); w is the load per unit of length; and t is the time. In Eq. 1 the damping force is considered to be a distributed force of $4\,\pi\,m\,\delta$ times the velocity of the beam at any point. The steady-state vibrations of a simply supported beam due to several different periodic forces are, for a uniformly distributed load $w \sin 2\,\pi\,f\,t$ (w positive downward),

$$y = \frac{4\,w\,l^4}{\pi^5\,E\,I}\left[\frac{1}{1^5}\,\beta_1 \sin\frac{\pi\,x}{l} + \frac{1}{3^5}\,\beta_3 \sin\frac{3\,\pi\,x}{l}\cdots + \frac{1}{n^5}\,\beta_n \sin\frac{n\,\pi\,x}{l}\right] \sin\,(2\,\pi\,f\,t - \alpha_n) \qquad (2a)$$

and

$$\theta = \frac{dy}{dx} = \frac{4\,w\,l^3}{\pi^4\,E\,I}\left[\frac{1}{1^4}\,\beta_1 \cos\frac{\pi\,x}{l} + \frac{1}{3^4}\,\beta_3 \cos\frac{3\,\pi\,x}{l}\cdots + \frac{1}{n^4}\,\beta_n \cos\frac{n\,\pi\,x}{l}\right] \sin\,(2\,\pi\,f\,t - \alpha_n) \qquad (2b)$$

in which β_n is the dynamic magnification factor; n is a series of integers, in this case add, 1, 3, 5 . . . ; f is the frequency of the periodic force; α is the phase angle; and l is span length.

For a concentrated load $P \sin 2\,\pi\,f\,t$ at the center (P positive downward):

$$y = \frac{2\,P\,l^3}{\pi^4\,E\,I}\left[\frac{1}{1^4}\,\beta_1 \sin\frac{\pi\,x}{l} - \frac{1}{3^4}\,\beta_3 \sin\frac{3\,\pi\,x}{l}\cdots \pm \frac{1}{n^4}\,\beta_n \sin\frac{n\,\pi\,x}{l}\right] \sin\,(2\,\pi\,f\,t - \alpha_n) \qquad (3a)$$

and

$$\theta = \frac{dy}{dx} = \frac{2\,P\,l^2}{\pi^3\,E\,I}\left[\frac{1}{1^3}\,\beta_1 \cos\frac{\pi\,x}{l} - \frac{1}{3^3}\,\beta_3 \cos\frac{3\,\pi\,x}{l}\cdots \pm \frac{1}{n^3}\,\beta_n \cos\frac{n\,\pi\,x}{l}\right] \sin\,(2\,\pi\,f\,t - \alpha_n) \qquad (3b)$$

For a concentrated load $P \sin 2\,\pi\,f\,t$ at a distance a from the left end (P positive downward):

$$y = \frac{2\,P\,l^3}{\pi^4\,E\,I}\left[\frac{1}{1^4}\,\beta_1 \sin\frac{\pi\,a}{l} \sin\frac{\pi\,x}{l} + \frac{1}{2^4}\,\beta_2 \sin\frac{2\,\pi\,a}{l} \sin\frac{2\,\pi\,x}{l}\cdots + \frac{1}{n^4}\,\beta_n \sin\frac{n\,\pi\,a}{l} \sin\frac{n\,\pi\,x}{l}\right] \sin\,(2\,\pi\,f\,t - \alpha_n) \qquad (4a)$$

and

$$\theta = \frac{dy}{dx} = \frac{2\,P\,l^2}{\pi^3\,E\,I}\left[\frac{1}{1^3}\beta_1 \sin\frac{\pi\,a}{l}\cos\frac{\pi\,x}{l} + \frac{1}{2^3}\beta_2 \sin\frac{2\,\pi\,a}{l}\cos\frac{2\,\pi\,x}{l}\cdots + \frac{1}{n^3}\beta_n \sin\frac{n\,\pi\,a}{l}\cos\frac{n\,\pi\,x}{l}\right]\sin\,(2\,\pi\,f\,t - \alpha_n)\,..(4b)$$

For a periodic moment $M \sin 2\,\pi\,f\,t$ at the left end (M positive clockwise):

$$y = \frac{2\,M\,l^2}{\pi^3\,E\,I}\left[\frac{1}{1^3}\beta_1 \sin\frac{\pi\,x}{l} + \frac{1}{2^3}\beta_2 \sin\frac{2\,\pi\,x}{l}\cdots + \frac{1}{n^3}\beta_n \sin\frac{n\,\pi\,x}{l}\right]\sin\,(2\,\pi\,f\,t - \alpha_n)\,..(5a)$$

and

$$\theta = \frac{dy}{dx} = \frac{2\,M\,l}{\pi^2\,E\,I}\left[\frac{1}{1^2}\beta_1 \cos\frac{\pi\,x}{l} + \frac{1}{2^2}\beta_2 \cos\frac{2\,\pi\,x}{l}\cdots + \frac{1}{n^2}\beta_n \cos\frac{n\,\pi\,x}{l}\right]\sin\,(2\,\pi\,f\,t - \alpha_n)\,..(5b)$$

In Eqs. 2 to 5, the dynamic magnification factor β_n, and the phase angle α_n, are to be defined as follows:

$$\beta_n = \frac{1}{\sqrt{\left(1 - \frac{f^2}{f^2_n}\right)^2 + \left(2\frac{\delta}{f_n} \times \frac{f}{f_n}\right)^2}} \,\ldots\ldots\ldots\ldots\ldots (6)$$

$$\tan \alpha_n = \frac{2\frac{\delta}{f_n} \times \frac{f}{f_n}}{1 - \frac{f^2}{f^2_n}} \,\ldots\ldots\ldots\ldots\ldots (7)$$

in which f_n, the natural frequency, is

$$f_n = \frac{1}{2\,\pi}\sqrt{\frac{n^4\,\pi^4\,E\,I}{m\,l^4}} \,\ldots\ldots\ldots\ldots\ldots (8)$$

The total deflection, y, in Eqs. 2 to 5 is the sum of several deflection curves, $\sin\frac{\pi\,x}{l}$, $\sin\frac{2\,\pi\,x}{l}\cdots\sin\frac{n\,\pi\,x}{l}$. These curves have the same shapes as the modes of free vibration for a simply supported beam. The dynamic magnification factor β_n determines the amplitude of each mode. A large amplitude occurs when the frequency of the periodic force f is equal to the natural frequency of one of the modes, $f_1, f_2, f_3\cdots f_n$. When $f = f_n$ the dynamic magnification factor β_n, for that mode, is a maximum and is large as compared with the values of β for the other modes. For this condition (called "synchronism") the total deflection, y, will consist of a large deflection having the shape of the mode for which $f = f_n$, plus some very minor deflections corresponding to the other modes.

The phase angle α_n is a convenient method of representing a time lag of from zero ($\alpha = 0°$) to one-half period ($\alpha = 180°$). The deflection y always

lags behind the periodic force by the phase angle α_n. In a study of vibrations the analyst is chiefly interested in the large deflection that occurs when the frequency of the periodic force is equal to one of the natural frequencies.

A study of the values of β_n, α_n, f_n, and δ shows that Eqs. 6 and 7 can be simplified considerably, when f equals f_1, f_2, f_3, or f_n; and $\delta < 0.1\, f_1$. Then

$$\left.\begin{aligned} \beta_n &= \frac{1}{2}\frac{f_n}{\delta} \text{ (for } f = f_n) \\ \beta_n &\cong \frac{1}{1 - \dfrac{f^2}{f^2_n}} \text{ (for } f \neq f_n) \end{aligned}\right\} \quad \text{(9)}$$

and

$$\left.\begin{aligned} \alpha_n &= 90° \text{ (for } f = f_n) \\ \alpha_n &= 0° \text{ or } 180° \text{ } (\tan \alpha < 0.05) \text{ (for } f \neq f_n) \end{aligned}\right\} \quad \text{(10)}$$

In this analysis of continuous frames the following conditions will be used:

$$\left.\begin{aligned} &\text{When } \frac{f}{f_n} < 1 \text{—} \\ &\qquad \alpha_n = 0° \text{ and } \sin(2\,\pi f t - \alpha_n) = \sin 2\,\pi f t \\ &\text{When } \frac{f}{f_n} = 1 \text{—} \\ &\qquad \alpha_n = 90° \text{ and } \sin(2\,\pi f t - \alpha_n) = \sin(2\,\pi f t - 90°) \\ &\text{When } \frac{f}{f_n} > 1 \text{—} \\ &\qquad \alpha_n = 180° \text{ and } \sin(2\,\pi f t - \alpha_n) = -\sin 2\,\pi f t \end{aligned}\right\} \quad \text{(11)}$$

2. General Method of Analysis

Large dynamic deflections in a continuous frame occur when the frequency of the periodic force coincides with the natural frequency of one of the modes of the loaded member. The members immediately adjacent to the loaded member must also have one natural frequency which coincides with the frequency of the periodic force. The continuous members must behave as simply supported members and the large deflections must coincide with the modes of a simply supported member.

The end slopes established by the periodic force are first determined. These will henceforth be called "free-end slopes." The geometric discontinuity that results from these free-end slopes is corrected by periodic end moments with the same frequency as the periodic force. The correction is made by a distribution procedure that corrects the end slopes joint by joint. The periodic moments applied to the ends meeting at a joint have a zero resultant. Thus, statics and dynamics are not affected while the geometry is being corrected. This distribution procedure requires the development of the stiffness of a member. The stiffness is defined as the periodic moment necessary to produce a slope of one (+1) at the end at which the moment is applied. Carry-over slopes are incurred and these can best be described subsequently in Section 4.

3. Free-End Slopes

Eqs. 2, 3, and 4 are used to compute the free-end slopes. Consider, for example, a periodic force $P \sin 2\pi f t$ at the center of a beam for which $f_1 = 5$, $f = f_1$, and $\delta = 0.1$. The free-end slopes are given by Eq. 3*b*, which yields

$$\theta_{x=0} = -\theta_{x=l} = \frac{2Pl^2}{\pi^3 EI}\left[\frac{1}{1^3}\beta_1 - \frac{1}{3^3}\beta_3 \cdots \pm \frac{1}{n^3}\beta_n\right]\sin(2\pi f t - \alpha_n) \quad \text{(12a)}$$

Since α_n can be taken as 0°, 90°, or 180°,

$$\theta_{x=0} = \frac{2Pl^2}{\pi^3 EI}\Bigg[\underbrace{\frac{1}{1^3}\beta_1 \sin(2\pi f t - 90°)}_{\left(\frac{f}{f_n} = 1 \text{ and } \alpha_n = 90°\right)} + \underbrace{\left(-\frac{1}{3^3}\beta_3 + \frac{1}{5^3}\beta_5 \cdots \pm \frac{1}{n^3}\beta_n\right)}_{\left(\frac{f}{f_n} < 1 \text{ and } \alpha_n = 0°\right)} \sin 2\pi f t\Bigg] \quad \text{(12b)}$$

For convenience in computing, let $\beta_n = \gamma_n + 1$, thus separating θ into static and dynamic parts:

$$\theta_{x=0} = \underbrace{\frac{2Pl^2}{\pi^3 EI}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \cdots \pm \frac{1}{n^3}\right]\sin 2\pi f t}_{\text{(Static)}}$$
$$+ \underbrace{\frac{2Pl^2}{\pi^3 EI} \times \frac{1}{1^3}\beta_1 \sin(2\pi f t - 90°)}_{\text{(Dynamic)}} - \underbrace{\frac{2Pl^2}{\pi^3 EI} \times \frac{1}{1^3}\sin 2\pi f t}_{\text{(Dynamic)}}$$
$$+ \underbrace{\frac{2Pl^2}{\pi^3 EI}\left[-\frac{1}{3^3}\gamma_3 + \frac{1}{5^3}\gamma_5 \cdots \pm \frac{1}{n^3}\gamma_n\right]\sin 2\pi f t}_{\text{(Dynamic)}} \quad \text{(12c)}$$

The series for the static part is equal to $\frac{1}{16}\frac{Pl^2}{EI}\sin 2\pi f t$; and data used in computing the dynamic part are presented in Table 1. Selecting values from this table for substitution in Eq. 12*c*,

$$\theta_{x=0} = \underbrace{\frac{1}{16}\frac{Pl^2}{EI}\sin 2\pi f t}_{\text{(Static)}}$$
$$+ \underbrace{\frac{2Pl^2}{\pi^3 EI} \times \frac{1}{1^3} 25.00 \sin(2\pi f t - 90°)}_{\text{(Dynamic)}} - \underbrace{\frac{2Pl^2}{\pi^3 EI} \times \frac{1}{1^3}\sin 2\pi f t}_{\text{(Dynamic)}}$$
$$+ \underbrace{\frac{2Pl^2}{\pi^3 EI}\left[-\frac{1}{3^3} \times 0.01250 + \frac{1}{5^3} \times 0.00160 \cdots\right]\sin 2\pi f t}_{\text{(Dynamic)}} \quad \text{(13)}$$

—from which

$$\theta_{x=0} = \frac{P\,l^2}{E\,I}\left[-\,0.00203 \sin 2\,\pi f\,t + 1.6126 \sin (2\,\pi f\,t - 90°)\right] \ldots (14a)$$

and

$$\theta_{x=0} = \frac{P\,l^2}{E\,I}\left[+\,0.00203 \sin 2\,\pi f\,t - 1.6126 \sin (2\,\pi f\,t - 90°)\right] \ldots (14b)$$

Fig. 1 represents the end slopes by means of vectors; they are shown in relation to the force P for $t = \frac{1}{4f}$ (one fourth of period) when P is a maximum downward. The projection of the slope vectors on the vertical axis will be equal to $\theta_{x=0}$ and $\theta_{x=l}$. These vectors rotate clockwise with an angular velocity of $2\,\pi f$.

TABLE 1.—DATA REQUIRED TO COMPUTE THE DYNAMIC PARTS OF EQ. 12c

n	f_n	f/f_n	β_n	γ_n	ᵃTan α_n
1	5	5/5 = 1	25.0	....	(90°)
2	20	5/20 = 4	1.06667	0.06667	0.00267
3	45	5/45 = 1/9	1.01250	0.01250	0.00050
4	80	5/80 = 1/16	1.00392	0.00392	0.000157
5	125	5/125 = 1/25	1.00160	0.00160	0.000064

ᵃ Values of tan γ_n are listed to show that $\alpha = 0°$ can be used when $f \neq f_n$.

In this position P is used as a reference for all slopes and end moments. Statics and dynamics have been satisfied for the periodic force $P \sin 2\,\pi f\,t$. The free-end slopes constitute a geometrical discontinuity with the adjacent members at the joints. This discontinuity must be corrected by periodic end moments.

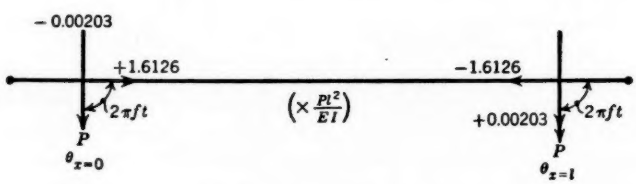


FIG. 1.—FREE-END SLOPES

4. STIFFNESS

The periodic moment required to produce a unit end slope is computed by Eq. 5. In this example the periodic moment $M \sin 2\,\pi f\,t$ is applied at the left end of a span for which $f_1 = 5, f = f_1$, and $\delta = 0.1$ (the same data as in Section 3); thus,

$$\theta_{x=0} = \frac{2\,M\,l}{\pi^2\,E\,I}\left[\frac{1}{1^2}\beta_1 + \frac{1}{2^2}\beta_2 \cdots \frac{1}{n^2}\beta_n\right] \sin (2\,\pi f\,t - \alpha_n) \ldots\ldots (15a)$$

and

$$\theta_{x=l} = \frac{2\,M\,l}{\pi^2\,E\,I}\left[-\,\frac{1}{1^2}\beta_1 + \frac{1}{2^2}\beta_2 \cdots \pm \frac{1}{n^2}\beta_n\right] \sin (2\,\pi f\,t - \alpha_n) \ldots (15b)$$

Since α_n can be taken as 0°, 90°, or 180°,

$$\theta_{x=0} = \frac{2\,M\,l}{\pi^2\,E\,I}\left[\underbrace{\frac{1}{1^2}\beta_1 \sin\,(2\,\pi\,f\,t - 90°)}_{\left(\frac{f}{f_n} = 1 \text{ and } \alpha = 90°\right)} + \underbrace{\left(\frac{1}{2^2}\beta_2 + \frac{1}{3^2}\beta_3 \cdots + \frac{1}{n^2}\beta_n\right)}_{\left(\frac{f}{f_n} < 1 \text{ and } \alpha = 0°\right)} \sin 2\,\pi\,f\,t\right] \quad (16)$$

Introducing $\beta_n = \gamma_n + 1$ as in Eq. 12*c*,

$$\theta_{x=0} = \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots \frac{1}{n^2}\right] \sin 2\,\pi\,f\,t}_{\text{(Static)}}$$
$$+ \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I} \times \frac{1}{1^2}\beta_1 \sin\,(2\,\pi\,f\,t - 90°)}_{\text{(Dynamic)}} - \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I} \times \frac{1}{1^2} \sin 2\,\pi\,f\,t}_{\text{(Dynamic)}}$$
$$+ \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I}\left[+\frac{1}{2^2}\gamma_2 + \frac{1}{3^2}\gamma_3 \cdots \frac{1}{n^2}\gamma_n\right] \sin 2\,\pi\,f\,t}_{\text{(Dynamic)}} \quad (17)$$

The series for the static part is equal to the static-end slope $\frac{1}{3}\frac{M\,l}{E\,I}\sin 2\,\pi\,f\,t$. As in Section 3, relating values from Table 1, for the dynamic parts—

$$\theta_{x=0} = \underbrace{\frac{1}{3} \times \frac{M\,l}{E\,I} \sin 2\,\pi\,f\,t}_{\text{(Static)}}$$
$$+ \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I} \times \frac{1}{1^2} \times 25.00 \sin\,(2\,\pi\,f\,t - 90°)}_{\text{(Dynamic)}} - \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I} \times \frac{1}{1^2} \times \sin 2\,\pi\,f\,t}_{\text{(Dynamic)}}$$
$$+ \underbrace{\frac{2\,M\,l}{\pi^2\,E\,I}\left[\frac{1}{2^2} \times 0.06667 + \frac{1}{3^2} \times 0.01250 + \frac{1}{4^2} \times 0.00392\right] \sin 2\,\pi\,f\,t}_{\text{(Dynamic)}} \quad (18)$$

—from which

$$\theta_{x=o} = \frac{M\,l}{E\,I}\left[0.134 \sin 2\,\pi\,f\,t + 5.07 \sin\,(2\,\pi\,f\,t - 90°)\right] \quad (19a)$$

and

$$\theta_{x=l} = \frac{M\,l}{E\,I}\left[0.0396 \sin 2\,\pi\,f\,t - 5.07 \sin\,(2\,\pi\,f\,t - 90°)\right] \quad (19b)$$

The end moment required to produce a unit end slope, defined as stiffness, S, is $\frac{1}{5.07} \times \frac{E\,I}{l}$ and for this moment θ becomes

$$\theta_{x=0} = [0.0265 \sin 2\,\pi\,f\,t + 1 \sin (2\,\pi\,f\,t - 90°)] \qquad (20a)$$

and

$$\theta_{x=l} = [0.00782 \sin 2\,\pi\,f\,t - 1 \sin (2\,\pi\,f\,t - 90°)] \qquad (20b)$$

In Eq. 20a the $+1$ is the required unit end slope, and the remaining numerical values in Eqs. 20, $+0.0265$, $+0.00782$, and the -1, are all carry-over slopes. Fig. 2 shows the vector diagram for the stiffness and carry-over slopes. Eqs. 20 are used also for a clockwise moment on the right end, but with $\theta_{x=0}$ and $\theta_{x=l}$ interchanged.

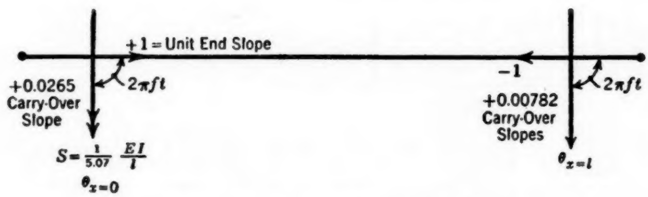


Fig. 2.—Stiffness and Carry-Over Slopes

A study of this analysis shows that, when the frequency of the periodic force is equal to one of the natural frequencies $f_1, f_2 \cdots f_n$, the stiffness is

$$S = \frac{\pi^2}{2} \times \frac{E\,I}{l} \times \frac{n^2}{1} \times \frac{1}{\beta_n} \qquad (21)$$

in which

$$\beta_n = \frac{1}{2} \times \frac{f_n}{\delta} = \frac{1}{2} \times \frac{n^2 f_1}{\delta} \qquad (22)$$

Substituting Eq. 22 in Eq. 21,

$$S = \frac{\pi^2\,E\,I}{l} \times \frac{\delta}{f_1} \text{ or } S \propto \frac{E\,I}{e} \times \frac{\delta}{f_1} \qquad (23)$$

The horizontal carry-over slope is always -1 for odd modes and $+1$ for even modes. The vertical carry-over slopes are plus or minus, but small as compared to 1.

5. Method for Correcting the Free-End Slopes by Periodic End Moments

Fig. 3(a) shows three members meeting at a joint with three end slopes $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$, which are not equal. Periodic moments with the same frequency as the periodic force $P \sin 2\,\pi\,f\,t$ are applied at the ends meeting at the joint. These moments must make the three horizontal components of the end slopes equal, and the moments must have a resultant of zero. The stiffness and carry-over slopes for each member are shown in Fig. 3(b). The relative stiff-

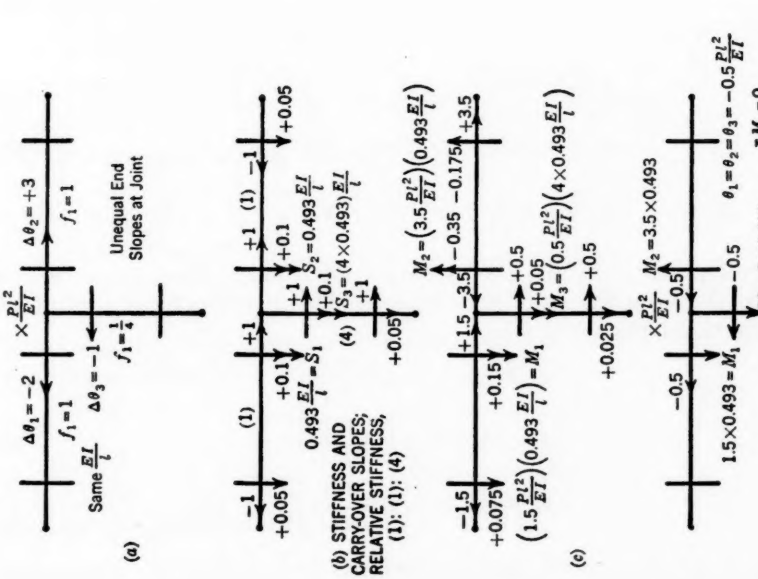


FIG. 3.—CORRECTION OF FREE-END SLOPES

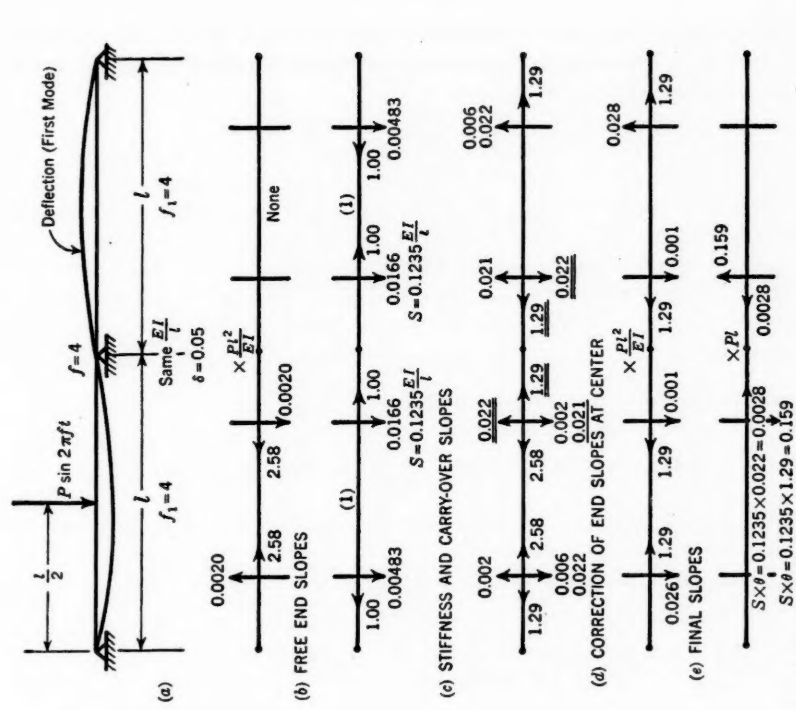


FIG. 4.—END MOMENTS AND SLOPES FOR TWO CONTINUOUS SPANS

nesses (assuming $E\,I/l$ the same for each member) are 1, 1, and 4. The formula for the end slope resulting from the end moments required to equalize the slopes for each member, is as follows:

$$\left.\begin{aligned} \theta_1 &= \frac{\Sigma S_n\,\Delta\theta_n}{\Sigma S_n} - \Delta\theta_1 \\ \theta_2 &= \frac{\Sigma S_n\,\Delta\theta_n}{\Sigma S_n} - \Delta\theta_2 \\ \theta_3 &= \frac{\Sigma S_n\,\Delta\theta_n}{\Sigma S_n} - \Delta\theta_3 \end{aligned}\right\}\qquad(24)$$

Then $\theta_n + \Delta\theta_n = \dfrac{\Sigma S_n\,\Delta\theta_n}{\Sigma S_n}$ (the same slope for each member). The end moments will have a zero resultant because—

$$\Sigma M_n = \Sigma S_n\,\theta_n = \left[\Sigma S_n\,\frac{\Sigma S_n\,\Delta\theta_n}{\Sigma S_n} - \Sigma S_n\,\Delta\theta_n\right] = 0\qquad(25)$$

For the example in Fig. 3,

$$\begin{aligned} \frac{\Sigma S_n\,\Delta\theta_n}{\Sigma S_n} &= \frac{S_1\,\Delta\theta_1 + S_2\,\Delta\theta_2 + S_3\,\Delta\theta_3}{S_1 + S_2 + S_3}\left(\frac{P\,l^2}{E\,I}\right) \\ &= \frac{[1 \times (-2)] + [1 \times (+3)] + [4 \times (-1)]}{[1 + 1 + 4]} \\ &= \frac{-3}{6} = -0.5\,\frac{P\,l^2}{E\,I}\qquad(26) \end{aligned}$$

Then

$$\left.\begin{aligned} \theta_1 &= -0.5 - (-2) = 1.5\,\frac{P\,l^2}{E\,I} \\ \theta_2 &= -0.5 - (+3) = -3.5\,\frac{P\,l^2}{E\,I} \\ \theta_3 &= -0.5 - (-1) = -0.5\,\frac{P\,l^2}{E\,I} \end{aligned}\right\}\qquad(27)$$

Multiply all quantities in Fig. 3(*b*) by the corresponding value θ_1, θ_2, and θ_3 and the results shown in Fig. 3(*c*) are obtained. The moments have a resultant of zero and the slopes are equal:

$$\left.\begin{aligned} \theta_1 + \Delta\theta_1 &= -0.5\,\frac{P\,l^2}{E\,I} \\ \theta_2 + \Delta\theta_2 &= -0.5\,\frac{P\,l^2}{E\,I} \\ \theta_3 + \Delta\theta_3 &= -0.5\,\frac{P\,l^2}{E\,I} \end{aligned}\right\}\qquad(28)$$

The vertical end slopes at the joint are corrected in the same way. The general procedure for the analysis consists of the following steps; illustrative examples will be explained: (1) Find free-end slopes; (2) find stiffness and carry-over

slopes for all members; (3) correct horizontal slopes and vertical slopes; (4) carry over the slopes; and (5) repeat steps (3) and (4).

6. Examples

The following examples illustrate the application of this analysis and demonstrate, to a limited extent, the nature of the behavior of continuous frames subjected to a periodic forced vibration. Fig. 4 shows a two-span continuous beam with a periodic force at the center of the left span. The free-end slopes, the stiffnesses, and carry-over slopes are given. Since the spans are identical they have a relative stiffness of 1:1. The analysis for the horizontal correction of the free-end slope at the center support is as follows:

$$\frac{\Sigma S_n \, \Delta\theta_n}{\Sigma S_n} = \frac{1 \times (-2.58)}{1+1} = -1.29$$

$$\theta_1 = -1.29 - (-2.58) = +1.29$$

$$\theta_2 = -1.29 - (0.0) = -1.29.$$

These values of θ_1 and θ_2 are shown with a double underline in Fig. 4(d). The carry-over slopes are obtained by multiplying the factors in the diagram by θ_1 and θ_2. For the left span the carry-over slopes are

$$\begin{aligned}
0.0166 &\times 1.29 = +0.021 \text{ (center)}\\
0.00483 &\times 1.29 = +0.006 \text{ (left end)}\\
-1.00 &\times 1.29 = -1.29 \text{ (left end)}
\end{aligned}$$

and for the right span,

$$\begin{aligned}
0.0166 &\times -1.29 = -0.021 \text{ (center)}\\
0.00483 &\times -1.29 = -0.006 \text{ (right end)}\\
-1.00 &\times -1.29 = +1.29 \text{ (right end).}
\end{aligned}$$

For the vertical correction:

$$\frac{\Sigma S_n \, \Delta\theta_n}{\Sigma S_n} = \frac{[1 \times (+0.023)] + [1 \times (-0.021)]}{1+1} = +0.001$$

$$\theta_1 = 0.001 - (+0.023) = -0.022$$

$$\theta_2 = 0.001 - (-0.021) = +0.022$$

These values of θ are shown with a double underline in Fig. 4(d). The carry-over slopes for the left span are, for the left end—

$$\begin{aligned}
0.0166 &\times -0.022 = -0.000365\\
0.00483 &\times -0.022 = -0.000106\\
-1.00 &\times -0.022 = +0.022
\end{aligned}$$

and, for the right span (right end)—

$$\begin{aligned}
0.0166 &\times 0.022 = +0.000365\\
0.00483 &\times 0.022 = +0.000106\\
-1.00 &\times 0.022 = -0.022.
\end{aligned}$$

Only the 0.022 quantities appear in Fig. 4, the other quantities being a remainder that is very small. There is a small difference of slope at the center equal to $2 \times 0.000365 = 0.00073$ which is not corrected.

The final slopes at the ends are obtained by adding, algebraically, the horizontal and vertical slope vectors. The final moments at the center are obtained by multiplying the θ-values at the center (these have a double underline in Fig. 4(d) by the stiffness, S. The frequency of the periodic force P is equal to the natural frequency of the first mode of these beams. Therefore the deflection has the shape of the first mode (see Fig. 4). There are some minor deflections of the shape of the other modes.

The bending moment and shear diagrams are derived from Eqs. 3 and 5 for the deflections. Eq. 5b, for a periodic moment on the left end of a simply supported beam, becomes the following equation for a periodic moment on the right end (M positive clockwise):

$$y = -\frac{2\,M\,l^2}{\pi^3\,E\,I}\left[\frac{1}{1^3}\beta_1 \sin\frac{\pi\,x}{l} - \frac{1}{2^3}\beta_2 \sin\frac{2\,\pi\,x}{l}\cdots \pm \frac{1}{n^3}\beta_n \sin\frac{n\,\pi\,x}{l}\right]\sin\,(2\,\pi\,f\,t - \alpha_n)\,..(29)$$

The bending moment for the left span can be written in the following manner, similar to the derivation for determining the free-end slopes, from Eq. 3a:

$$M_P = \underbrace{\frac{2\,P\,l}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi\,x}{l} - \frac{1}{3^2}\sin\frac{3\,\pi\,x}{l}\cdots \pm \frac{1}{n^2}\sin\frac{n\,\pi\,x}{l}\right]\sin\,(2\,\pi\,f\,t)}_{\text{(Static)}}$$

$$+ \underbrace{\frac{2\,P\,l}{\pi^2}\frac{1}{1^2}\beta_1 \sin\frac{\pi\,x}{l}\sin\,(2\,\pi\,f\,t - 90°)}_{\text{(Dynamic)}} - \underbrace{\frac{2\,P\,l}{\pi^2}\frac{1}{1^2}\sin\frac{\pi\,x}{l}\sin 2\,\pi\,f\,t}_{\text{(Dynamic)}}$$

$$+ \underbrace{\frac{2\,P\,l}{\pi^2}\left[-\frac{1}{3^2}\gamma_3 \sin\frac{3\,\pi\,x}{l} + \frac{1}{5^2}\gamma_5 \sin\frac{5\,\pi\,x}{l}\cdots \pm \frac{1}{n^2}\gamma_n \sin\frac{n\,\pi\,x}{l}\right]\sin 2\,\pi\,f\,t}_{\text{(Dynamic)}}\,..(30)$$

and from Eq. 29, for a periodic moment, M_m, on the right end:

$$M_m = -\underbrace{\frac{2\,M}{\pi}\left[\frac{1}{1}\sin\frac{\pi\,x}{l} - \frac{1}{2}\sin\frac{2\,\pi\,x}{l}\cdots \pm \frac{1}{n}\sin\frac{n\,\pi\,x}{l}\right]\sin 2\,\pi\,f\,t}_{\text{(Static)}}$$

$$- \underbrace{\frac{2\,M}{\pi}\frac{1}{1}\beta_1 \sin\frac{\pi\,x}{l}\sin\,(2\,\pi\,f\,t - 90°)}_{\text{(Dynamic)}} - \underbrace{\frac{2\,M}{\pi}\left(-\frac{1}{1}\sin\frac{\pi\,x}{l}\right)\sin 2\,\pi\,f\,t}_{\text{(Dynamic)}}$$

$$- \underbrace{\frac{2\,M}{\pi}\left[-\frac{1}{2}\gamma_2 \sin\frac{2\,\pi\,x}{l} + \frac{1}{3}\gamma_3 \sin\frac{3\,\pi\,x}{l}\cdots\frac{1}{n}\gamma_n \sin\frac{n\,\pi\,x}{l}\right]\sin 2\,\pi\,f\,t}_{\text{(Dynamic)}}\,..(31)$$

In this latter equation $M = 0.159\,P\,l$ (from Fig. 4).

The bending moment diagram for part of Eq. 30 is a diagram with the same shape as that of the diagram for the static simple-beam bending moment for concentrated load, P. The ordinates vary, periodically, with the bending moment under the load equal to $\frac{P\,l}{4} \sin 2\,\pi\,f\,t$.

The static part of Eq. 31 is a diagram with the same shape as that of the diagram for the static bending moment with a moment on the end. The ordinates vary periodically with the end moment, which is $0.159\,P\,l \sin 2\,\pi\,f\,t$. The bending-moment diagrams for the terms marked "dynamic" are sine curves, as shown, with ordinates that vary periodically.

The terms that contain β_1 are so much larger than all the other terms that only these need be considered. There are two terms which contain β_1:

$$\frac{2\,P\,l}{\pi^2} \times \frac{1}{1^2}\,\beta_1 \frac{\sin \pi\,x}{l} \sin\,(2\,\pi\,f\,t - 90°)$$

and

$$-\frac{2\,M}{\pi} \times \frac{1}{1}\,\beta_1 \sin \frac{\pi\,x}{l} \sin\,(2\,\pi\,f\,t - 90°).$$

The bending-moment diagram for the left span is essentially a sine curve with ordinates that vary periodically. The center ordinate is $\left[\left(\frac{2\,P\,l}{\pi^2} \times \frac{1}{1^2}\,\beta_1\right) - \left(\frac{2\,M}{\pi} \times \frac{1}{1}\,\beta_1\right)\right] \sin\,(2\,\pi\,f\,t - 90°)$. Since $\beta_1 = \frac{1}{2}\frac{f_1}{\delta} = \frac{1}{2} \times \frac{4}{0.05} = 40$ and $M = 0.159\,P\,l$; the center ordinate is $(8.10\,P\,l - 4.04\,P\,l) \sin\,(2\,\pi\,f\,t - 90°)$ or $4.06\,P\,l \sin\,(2\,\pi\,f\,t - 90°)$.

A comparison of the dynamic and static moments is interesting. Over the center support the dynamic is only 1.5 times the static, whereas, at the middle of the left span, the dynamic is twenty times the static. Therefore, as previously stated, the continuous beam behaves as two simply supported beams.

The shear diagrams are obtained from Eqs. 30 and 31. As before the congruous shear diagram for the static part of the equation for the bending moment M_P derived from Eq. 3 is a diagram with the same shape as that of the diagram for the simple beam shear diagram for a concentrated load, P. The ordinates vary periodically, equal to $\frac{P}{2} \sin 2\,\pi\,f\,t$. The congruous shear diagram for the static part of Eq. 31 is a diagram with the same shape as that of the diagram for the static shear diagram for a moment on the end. The ordinates vary periodically, equal to $-0.159\,P \sin 2\,\pi\,f\,t$. The shear diagrams for the terms marked "dynamic" are obtained by differentiation. These shear diagrams are cosine curves with ordinates that vary periodically.

The terms that contain β_1 are so much larger than all the other terms that only these need be considered. These same terms for the bending moment, on differentiation, become

$$\frac{2\,P}{\pi}\,\frac{1}{1}\,\beta_1 \cos \frac{\pi\,x}{l} \sin\,(2\,\pi\,f\,t - 90°)$$

and

$$-\frac{2M}{l}\beta_1 \cos\frac{\pi x}{l} \sin(2\pi f t - 90°).$$

The shear diagram, then, is a cosine curve with a value at the left end of

$$12.76\, P \sin(2\pi f t - 90°).$$

Fig. 5 shows the results of several analyses of the previous example, with some variations. The results shown in Fig. 5(a) are for the same data as those in

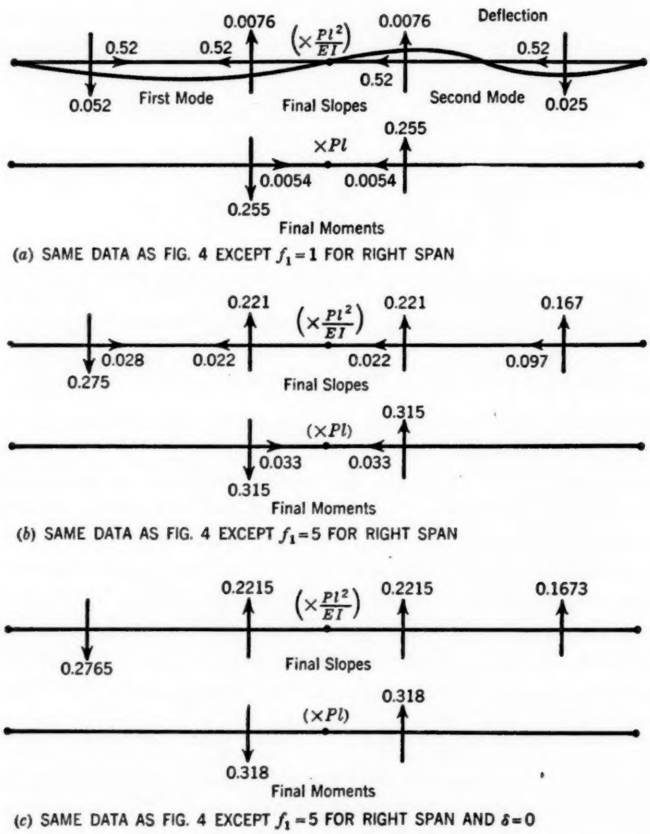


Fig. 5.—End Moments and Slopes for Two Continuous Spans

Fig. 4 with the exception that the fundamental frequency of the right span was changed from 4 to 1. Then $f_2 = 4$ for the right span and the second mode was excited by the periodic force. The first mode of the left span and the second mode of the right span comprise the deflection (Fig. 5(a)). This change of frequency increased the stiffness of the right span and this, in turn, reduced the end slope (and deflection) from that obtained in Fig. 4; and it increased the

moment at the center support. For synchronous conditions the stiffness varies as follows:

$$S \propto \frac{n^2 E I}{l \beta_n} \qquad (32a)$$

Since $\beta_n = \frac{1}{2}\frac{f_n}{\delta} = \frac{1}{2}\frac{n^2 f_1}{\delta}$ and $f_1 = \frac{\pi^2}{2 l^2}\sqrt{\frac{E I}{m}}$,

$$S \propto \frac{E I \delta}{l f_1} \propto l \delta \sqrt{m E I} \qquad (32b)$$

In this example the change in f_1 (from 4 to 1), without any change in the other data, resulted in a four-fold increase of stiffness for the right span and the relative stiffness changed from 1:1 to 1:4. When the stiffness was 1:1, the end slope was reduced to about one half whereas, for a stiffness ratio of 1:4 the end slope is reduced to about one fifth.

For the results shown in Fig. 5(*b*) the frequency of the right span was changed to 5, all other data remaining the same as those shown in Fig. 4. The frequency of the periodic force P does not coincide with the natural frequency of any mode of the right span. This analysis shows a very small end slope at the center; that is, the left span behaves as if it were fixed at the right end. The bending moment at the center support is large—approximately 3.5 times the static moment at this point. The example shows the marked effect of the relation between the frequency of the periodic force and the natural frequency of the members.

Fig. 5(*c*) illustrates the same problem as Fig. 5(*b*) but with no damping, and the results are of particular interest. It is evident from a comparison of Figs. 5(*b*) and 5(*c*) that the damping has little influence in this problem. The important factor influencing the dynamic behavior is the change of frequency of the right span from 4 to 5 so that $f_n \neq f$. It is the writer's opinion that damping plays a minor role in the steady-state vibration of structures, and that, on the other hand, the factors affecting the natural frequency of members play a major role.

Fig. 6(*a*) shows a structure consisting of three continuous spans with two columns. The members are all equal and have the same properties as the beams in Fig. 4. Fig. 6(*b*) shows the first horizontal and vertical correction of the end slopes at the left joint and at the right joint. These corrections are continued in Fig. 6(*d*), in which the values of θ_n are all double-underlined and the total change of slope and $\Sigma\theta_n$ are given. The final end slopes are shown in Fig. 6(*c*), which also shows the final end moments, obtained by multiplying $\Sigma\theta_n$ by S_n.

A comparison of these results with those shown in Fig. 4 is interesting. Increasing the number of members decreases the end slopes in approximately direct proportion to the number of members. In Fig. 4 there are two beams and the final end slope is about one half of 2.58; in Fig. 6(*c*) there are five beams and the final end slope is about one fifth of 2.58. This reduction of end slopes and deflections with the increase in the number of members is the result of damping. The shape of the deflection curve of all members is that of the first mode (see Fig. 5). Approximately the same results would have been obtained by con-

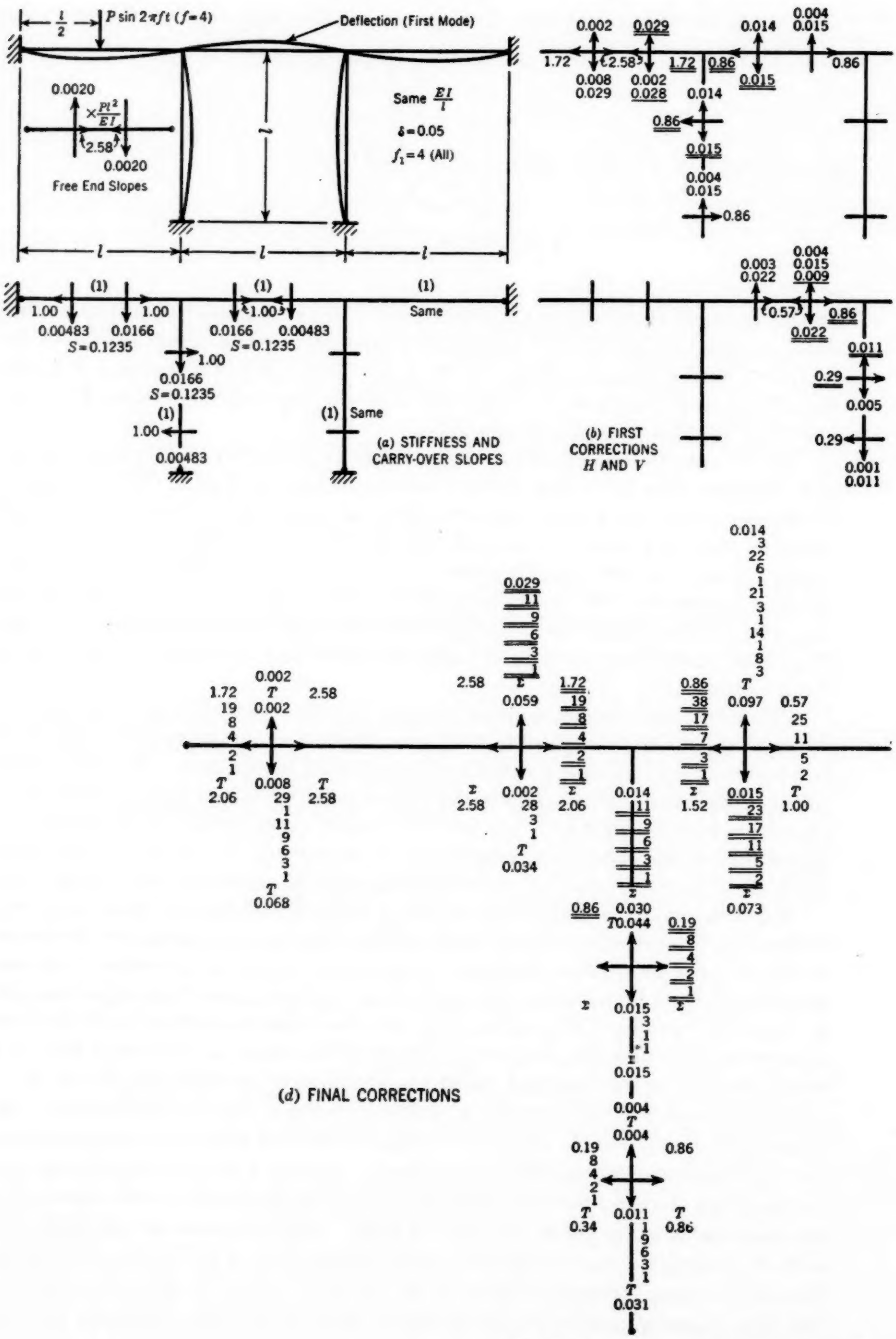
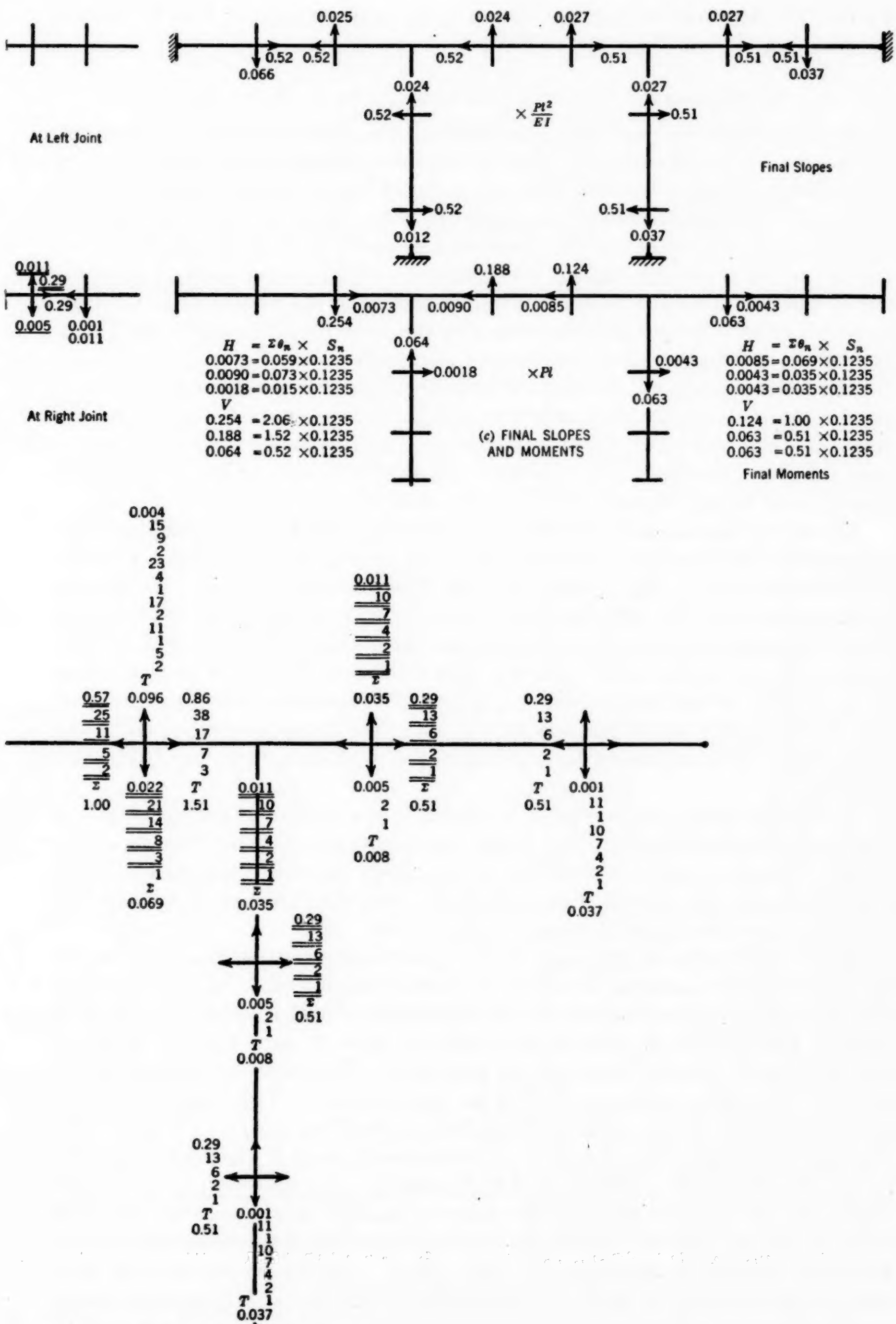


FIG. 6.—END MOMENTS AND SLOPES FOR TH



REE CONTINUOUS SPANS WITH TWO COLUMNS

sidering the structure in Fig. 6(a) as a simply supported beam with five times the damping (δ = 0.25 instead of 0.05).

7. Supplementary Procedure for Lateral Displacement

In the analysis that has been presented, all the joints were fixed against displacement. In most cases the effect of joint displacement is negligible, but it is possible to assign conditions that will magnify lateral displacement of joints. In Fig. 7, which illustrates displacement of joints, conditions have been chosen which magnify this effect. The bent has a periodic force at the quarter point with a frequency equal to the natural frequency of the second mode of the horizontal member and equal to the natural frequency of the first mode of the legs. Under these conditions both legs deflect in the same direction (see Fig. 7) and, as a result, will have a large unbalanced horizontal shear.

The following procedure was used in analyzing the effect of lateral displacement. An analysis was first made with the joints fixed against lateral displacement, and the restraining force was found. Then an analysis was made for a periodic lateral displacement in phase with force P, and the force necessary to produce this lateral displacement was determined.

The lateral displacement caused by a force equal and opposite to the restraining force can be found by proportion and by changing the phase of the displacement relative to P. Fig. 7 shows the results of such an analysis. On the left is the analysis for the bent restrained from lateral displacement. The figure shows the moment at the top of the legs and the shears. The sum of the shears is the total restraining force. On the right are shown the results of an analysis for a periodic lateral displacement in phase with the force P. The moments and shears at the top of the legs are shown. The total force is equal to the shear plus a force necessary to give a periodic displacement to the horizontal member.

A periodic lateral displacement, as shown at the bottom of the figure, would require a force equal and opposite to the restraining force. Therefore an analysis that includes lateral displacement of the joints involves the addition of the effect of a periodic lateral displacement to the effect of the periodic forced vibration restrained against lateral displacement.

In the application of this analysis to engineering structures it may be necessary to take into account the effect of a concentrated mass on the structure. Fig. 8(a) shows a beam with a mass $[M]$ associated with a periodic force $P \sin 2\pi f t$. First, make an analysis for a periodic force $P' \sin 2\pi f t$, the force between the mass and the beam, as yet unknown. The rotating vector diagram shows P', α', and y_a; and a vector $[M]\ddot{y}_a$ opposite to y_a. This latter vector represents the force on the mass $[M]$ and is equal to $P \sin 2\pi f t - P' \sin 2\pi f t$. The known periodic force $P \sin 2\pi f t$ is the resultant of $P' \sin 2\pi f t$ and $[M]\ddot{y}_a$, and determines the magnitude of P', y_a, and α. If a mass $[M_b]$ is at a point other than that where the periodic force is located, as shown Fig. 8(b), first make an analysis for the force $P \sin 2\pi f t$ and determine the deflection and acceleration of point B neglecting the mass $[M_b]$. Make a second analysis for a force $P' \sin 2\pi f t$ at point B, the force between the mass $[M_b]$ and the beam, as yet unknown. Since a reciprocal relation exists between points A and B,

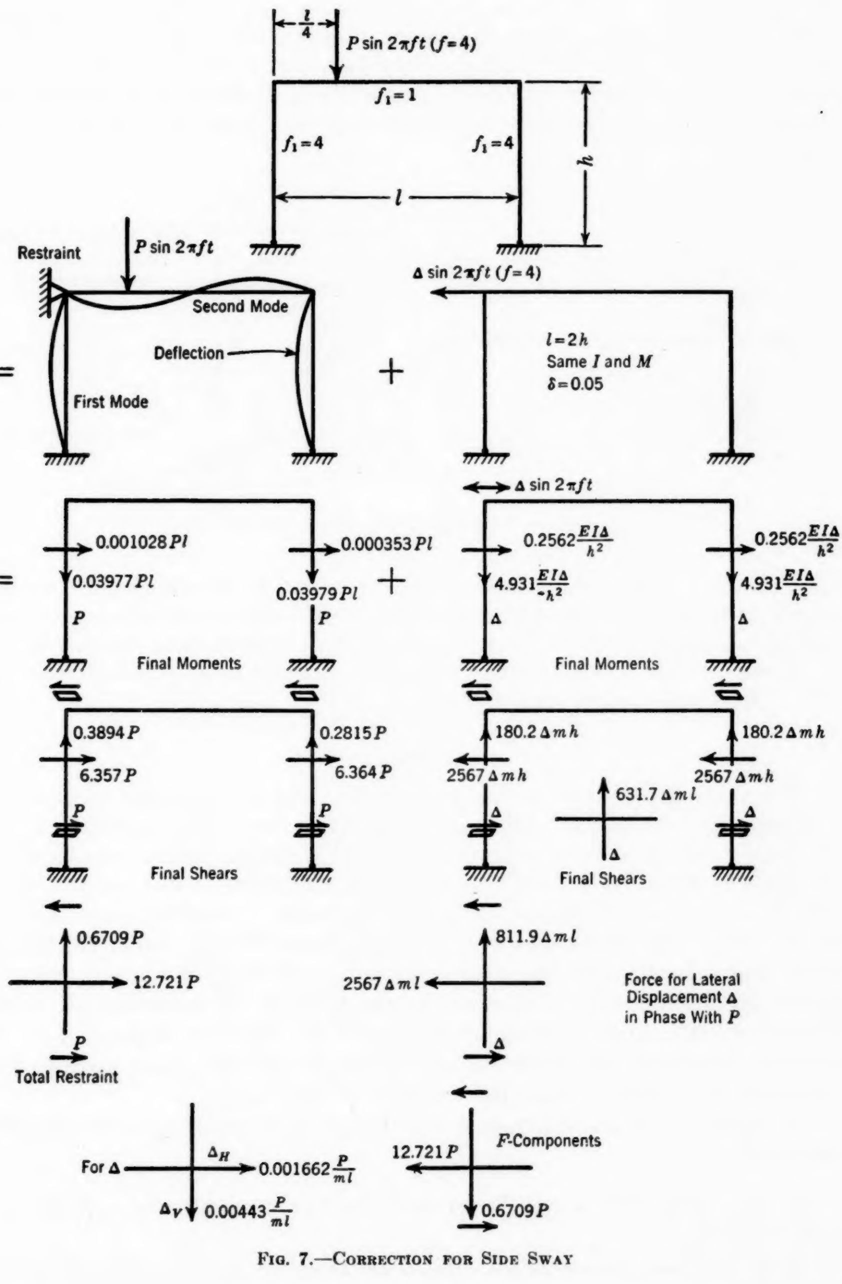


Fig. 7.—Correction for Side Sway

Fig. 8(*b*), the first analysis will give the relative proportions of these vectors and the phase angles. Combine the two acceleration vectors at point B—$\ddot{y}_{ba}$ due to P at point A, and $\ddot{y}_{bb}$ due to P' at point A, so that the resultant acceleration $\ddot{y}_b$ is in line with P' and $P' = M_b\ddot{y}_b$. The force on the beam, P', will then be equal and opposite to the force on the mass $[M_b]\ddot{y}_b$. The total deflection will be the vector sum of that due to P and that due to P', the phase relation with P as shown.

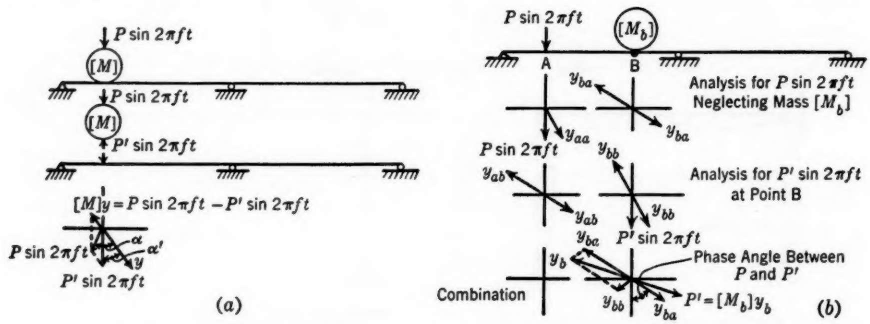


FIG. 8.—EFFECT OF CONCENTRATED MASS

The effect of several periodic forces may need to be considered. If the several forces have the same frequency, the vector sum of the several solutions with proper regard to phase relations will result in a steady-state periodic forced vibration. If the several forces have different frequencies separate analyses can be made and added vectorially.

8. SUMMARY

The analysis described in this paper is designed as a practical method for studying the vibration of civil engineering structures. The assumptions are those commonly used in structural analysis and in the analysis of the vibrations of simple systems. The distribution procedure adapted to this analysis is an analytical tool familiar to most structural engineers. Distribution of the end slopes rather than of the end moments was chosen because initial conditions more closely approximate the final conditions, and therefore there is less correction by distribution. Experience in the behavior of structures for static loads is of little value to the understanding of the vibration of structures. A practical analytical procedure has been lacking, and this paper supplies the necessary equipment to study the vibration of structures.

In conclusion a few statements will be added to supplement the analysis described:

1. This method of analysis is not restricted to the conditions $f = f_1, f_2 \cdots f_n$ and $\delta < 0.1 f_1$;

2. Several periodic forces with the same frequency can be analyzed simultaneously;

3. Several periodic forces at different frequencies must be analyzed separately and the solutions added;

4. The use of modified stiffnesses is of particular value because of the large carry-over slopes; and

5. Free vibrations with velocity damping can be added to the steady-state vibrations to obtain the initial vibrations. Consequently an analysis for transients can be made.